

# Some remarks on morphisms between Fano threefolds

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Some twenty-five years ago, Iskovskih classified the smooth complex Fano threefolds with Picard number one. Apart from  $\mathbb{P}^3$  and the quadric, his list includes 5 families of Fano varieties of index two and 11 families of varieties of index one (for index one threefolds, the cube of the anticanonical divisor takes all even values from 2 to 22, except 20). Recently, the author ([A]) and C. Schuhmann ([S]) made some efforts to classify the morphisms between such Fano threefolds, the starting point being a question of Peternell: let  $f : X \rightarrow Y$  be a non-trivial morphism between Fano varieties with Picard number one, is it then true that the index of  $X$  does not exceed the index of  $Y$ ?

In particular, Schuhmann ([S]) proved that there are no morphisms from index-two to index-one threefolds, and that any morphism between index-two threefolds is an isomorphism (under certain mild additional hypotheses, some of which were handled later in [A], [IS]). As for morphisms from index-one to index-two Fano threefolds, such morphisms do exist: an index-two threefold has a double covering (branched along an anticanonical divisor) which is of index one. It is therefore natural to ask if every morphism from index-one Fano threefold  $X$  with Picard number one to index-two Fano threefold  $Y$  with Picard number one is a double covering. In [A], I proved a theorem (Theorem 3.1) indicating that the answer should be yes, however not settling the question completely. The essential problem was that the methods of [A] would never work for  $Y = V_5$ , the linear section of the Grassmannian  $G(1, 4)$  in the Plücker embedding (all smooth three-dimensional linear sections of  $G(1, 4)$  are isomorphic). Though there are several ways to obtain bounds for the degree of a morphism between Fano threefolds with second Betti number one ([HM], [A]), these bounds are still too rough for our purpose.

This paper is an attempt to handle this problem. The main result is the following

**Theorem** *Let  $X$  be a smooth complex Fano threefold of index one and such that  $\text{Pic}(X) = \mathbb{Z}$ . Suppose moreover that  $X$  is anticanonically embedded. Let  $f : X \rightarrow V_5$  be a non-trivial morphism. Then  $X$  is of degree 10 (“ $X$  is of type  $V_{10}$ ”) and  $f$  is a double covering. In other words,  $X$  is a hyperquadric section of a cone over  $V_5$  in  $\mathbb{P}^7$ .*

I believe that the extra assumption made on  $X$  is purely technical and can be ruled out if one refines the arguments below. This assumption excludes two families of Fano threefolds: sextic double solids and double coverings of the quadric branched along a hyperquartic section. A smooth anticanonically embedded Fano threefold of index one and Picard number one is sometimes called *a prime Fano threefold*. We shall also call it thus throughout this paper.

## 1. Preliminaries: the geometry of $V_5$

Let us recall some more or less classical facts on the threefold  $V_5 \subset \mathbb{P}^6$ , most of which can be found in [I] or [FN]. First of all, as any Fano threefold of index two and Picard number 1, it has a two-dimensional family of lines. A general line has trivial normal bundle (call it a  $(0,0)$ -line), whereas there is a one-dimensional subfamily of lines with normal bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$  (call them  $(-1,1)$ -lines). The Hilbert scheme of lines on  $V_5$  is isomorphic to  $\mathbb{P}^2$ , the curve of  $(-1,1)$ -lines is a conic in this  $\mathbb{P}^2$ , and there are 3 lines through a general point of  $V_5$ . More precisely, the  $(-1,1)$ -lines form the tangent surface  $D$  to a rational normal sextic  $B$  on  $V_5$  (in particular, they never intersect), and there are three lines through any point away from  $D$ , two lines through a point on  $D$  but not on  $B$ , and one line through a point of  $B$ . The surface  $D$  is of degree 10, thus a hyperquadric section of  $V_5$ .

We shall denote by  $U$  resp.  $Q$  the restriction to  $V_5$  of the universal bundle  $U_G$  resp. the universal quotient bundle  $Q_G$  on the Grassmannian  $G(1,4)$ . The cohomology groups related to those bundles are computed starting from the cohomologies of vector bundles on the Grassmannian. In particular the bundles  $U$  and  $Q$  remain stable.

We shall also use the following result from [S]: let  $X$  be a prime Fano threefold, and let  $f : X \rightarrow V_5$  be a finite morphism. Let  $m$  be such that  $f^*\mathcal{O}_{V_5}(1) = \mathcal{O}_X(m)$ . Then the inverse image of a general line consists of  $\frac{m^2 \deg(X)}{10}$  disjoint conics; in general, if one replaces  $V_5$  by another Fano threefold  $Y$  of index two with Picard number one, the inverse image of a general line shall consist of  $\frac{m^2 \deg(X)}{2 \deg(Y)}$  disjoint conics. Here by  $\deg(Y)$  we mean the self-intersection number of the ample generator of  $\text{Pic}(Y)$ .

Our starting point is the observation that the inverse image of a  $(-1,1)$ -line must be connected. This will be the main result of this paragraph.

The Schubert cycles of type  $\sigma_{1,1}$ , which are sets of points of  $G(1,4)$  corresponding to lines lying in a fixed hyperplane, and are also characterized as zero-loci of sections of the bundle dual to the universal, are 4-dimensional quadrics in the Plücker  $G(1,4)$ , so each of them intersects  $V_5$  along a conic. Conversely, every smooth conic on  $V_5$  is an intersection with such a Schubert cycle. Indeed, every conics on a Grassmannian is obviously contained in some  $G(1,3)$ ; and if this conic is strictly contained in  $G(1,3) \cap V_5$ , then  $G(1,3) \cap V_5$  is a surface, so the bundle  $U^*$  has a section vanishing along a surface; but this contradicts the stability of  $U^*$ .

The same is (by the same argument) true for pairs of intersecting lines on  $V_5$ .

Moreover, the correspondence between the Schubert cycles and the conics is one-to-one (it is induced by the restriction map on the global sections  $H^0(G(1, 4), U_G^*) \rightarrow H^0(V_5, U^*)$  which is an isomorphism).

Let us show that among these conics, there is a one-dimensional family of double lines.

**Proposition 1.1** *Fix an embedding  $V_5 \subset G(1, 4) \subset \mathbb{P}^9$ . There is a one-dimensional family of Schubert cycles  $\Sigma_t$  such that for each  $t$ , the intersection of  $V_5$  and  $\Sigma_t$  is a (double) line. Moreover, lines on  $V_5$  which are obtained as a set-theoretic intersection with a Schubert cycle of type  $\sigma_{1,1}$ , are exactly  $(-1, 1)$ -lines.*

*Proof:* The three-dimensional linear sections of  $G(1, 4)$  in the Plücker embedding are parametrized by the Grassmann variety  $G(6, 9)$ ; let, for  $P \in G(6, 9)$ ,  $V_P$  denote the intersection of  $G(1, 4)$  with the corresponding linear subspace (which we will denote also by  $P$ ). The Schubert cycles are parametrized by  $G(3, 4) = \mathbb{P}^4$ ; likewise, denote by  $\Sigma_t$  the Schubert cycle corresponding to  $t \in \mathbb{P}^4$ . Consider the following incidence subvariety  $\mathcal{I} \subset G(6, 9) \times \mathbb{P}^4$ :

$$\mathcal{I} = \{(P, t) \in G(6, 9) \times \mathbb{P}^4 \mid V_P \cap \Sigma_t \text{ is a line}\}.$$

The fiber  $\mathcal{I}_t$  of  $\mathcal{I}$  over any  $t \in \mathbb{P}^4$  parametrizes the six-dimensional subspaces  $P$  of  $\mathbb{P}^9$  intersecting  $\Sigma_t$  along a line.  $\Sigma_t$  is a quadric in  $\mathbb{P}^5 \subset \mathbb{P}^9$ , and  $P$  intersects it along a line  $l$  if and only if the plane  $H = P \cap \mathbb{P}^5$  is tangent to  $\Sigma_t$  along  $l$ , i.e. lies in every  $T_x \Sigma_t$ ,  $x \in l$ . The intersection of all tangent spaces to  $\Sigma_t \subset \mathbb{P}^5$  along  $l$  is a three-dimensional projective space (the tangent spaces form a pencil of hyperplanes in  $\mathbb{P}^5$ , because  $\Sigma_t$  is a quadric). This means that for every  $l$ , the planes tangent to  $\Sigma_t$  along  $l$  form a one-dimensional family. The family of lines on a 4-dimensional quadric ( $= G(1, 3)$ ) is a 5-dimensional flag variety, so the planes in  $\mathbb{P}^5$  tangent to  $\Sigma_t$  along a line are parametrized by a six-dimensional irreducible variety (a  $\mathbb{P}^1$ -bundle over a flag variety). This implies that  $\mathcal{I}_t$  is irreducible of codimension 3 in  $G(6, 9)$ , so  $\mathcal{I}$  is irreducible of codimension 3 in  $G(6, 9) \times \mathbb{P}^4$ .

We must show that the first projection  $p_1 : \mathcal{I} \rightarrow G(6, 9)$  is surjective and its general fiber is of dimension one. First of all, remark that there are points  $P$  in the image of  $p_1$  such that the corresponding  $V_P$  is smooth (so, is a  $V_5$ ). Indeed, fix, as above,  $\Sigma_t$ ,  $l \subset \Sigma_t$ ,  $H$  a plane in  $\mathbb{P}^5 = \langle \Sigma_t \rangle$  such that  $H \cap \Sigma_t = l$ ; the remark will follow if we show that for a general  $\mathbb{P}^6 = P \subset \mathbb{P}^9$  containing  $H$ ,  $G(1, 4) \cap P$  is smooth. We have  $H \cap G(1, 4) = H \cap \Sigma_t = l$  (because  $G(1, 4) \cap \langle \Sigma_t \rangle = \Sigma_t$ ), so the smoothness away from  $l$  is obvious, and one checks, again by standart dimension count, that for  $x \in l$ , the set  $A_x = \{P \mid H \subset P, G(1, 4) \cap P \text{ is singular at } x\}$  is of codimension two in the space of all  $P$ 's containing  $H$ . Therefore for  $P$  general in the image of  $p_1$ ,  $V_P$  is smooth.

It is clear that if a smooth  $V_P = G(1, 4) \cap P$  is such that  $V_P \cap \Sigma_t = l$ , then the corresponding plane  $H$  is tangent along  $l$  not only to  $\Sigma_t$ , but also to  $V_P$ . Thus the normal bundle  $N_{l, V_P}$  has a subbundle  $N_{l, H}$  of degree 1, and so  $l$  is of type  $(-1, 1)$

on  $V_P$ . Since we have only one-dimensional family of  $(-1, 1)$ -lines on a smooth  $V_P$ , we deduce that a fiber of  $p_1$  over a point  $P$  such that  $V_P$  is smooth, is at most one-dimensional. The irreducibility of  $\mathcal{I}$  now implies that  $p_1$  is surjective and its general fiber is of dimension one. This proves the Proposition.

Let us now recall the following result of Debarre ([D], partial case of Théorème 8.1, Exemple 8.2 (3)):

*Let  $X$  be an irreducible projective variety, and let  $f : X \rightarrow G(d, n)$  be a morphism. Let  $\Sigma$  be a Schubert cycle of type  $\sigma_m$ . If in the cohomologies of  $G(d, n)$ ,  $[f(X)] \cdot \sigma_{m+1} \neq 0$ , then  $f^{-1}(\Sigma)$  is connected.*

Let  $X$  be an irreducible projective variety and  $f : X \rightarrow V_5$  be a surjective morphism. Composing with the embedding  $i : V_5 \rightarrow G(1, 4)$ , we can view  $f$  as a morphism to  $G(1, 4)$ . The Schubert cycles that we have just considered are of type  $\sigma_{1,1}$  and not  $\sigma_2$ , however, passing to the dual projective space, we arrive at the situation of Debarre's theorem and get the following

**Corollary 1.2** *For  $l$  a  $(-1, 1)$ -line on  $V_5$ , any irreducible projective variety  $X$  and a surjective morphism  $f : X \rightarrow V_5$ ,  $f^{-1}(l)$  is connected.*

**Remark 1.3** If we knew that the inverse image of a *general* line is always connected, this would immediately solve our problem; indeed, for a Fano threefold  $X$  of index and Picard number one, the equality  $\frac{m^2 \deg(X)}{10} = 1$  implies that  $m = 1$ ,  $\deg(X) = 10$  and  $f$  is a double covering. However, as shows an example of Peternell and Sommese, this is false in general, even if one supposes that  $X$  is a Fano threefold. In the example of [PS],  $X$  is the universal family of lines on  $V_5$ , which turns out to be a Fano threefold (of Picard number two, of course), and  $f$  is the natural triple covering. The inverse image of a general line has two connected components.

**Remark 1.4** One can ask if there is a similar connectedness statement for other Fano threefolds of Picard number one and index two. Recall that these are the following: intersection of two quadrics in  $\mathbb{P}^5$ ; cubic in  $\mathbb{P}^4$ ; double covering of  $\mathbb{P}^3$  branched in a quartic; double covering of the cone over Veronese surface branched in a hypercubic section.

Smooth quadrics in  $\mathbb{P}^5$  are Grassmannians  $G(1, 3)$ , and a smooth intersection of two quadrics in  $\mathbb{P}^5$  is a quadric line complex. It is classically known (see [GH], Chapter 6) that on a quadric line complex, there is a finite (and non-zero) number of lines obtained as set-theoretic intersection with a plane in  $G(1, 3)$ . These lines are obviously  $(-1, 1)$ -lines, since the corresponding plane is tangent to the quadric line complex along this line. Our intersection of two quadrics is contained in a pencil of such Grassmannians, so there is a one-dimensional family of lines on it such that each line is the intersection with a plane lying on some Grassmannian of the pencil. The curve of  $(-1, 1)$ -lines is irreducible (it follows from the results in [GH], Chapter 6, that it is smooth and that it is an ample divisor on the Fano surface of lines, in

particular, it is connected). Thus it is just the closure of that family. So that it follows again from Debarre's paper that the inverse image of a general  $(-1, 1)$ -line is connected.

As for the cubic, even if such a connectedness statement could hold, it would not, as far as I see, follow from any well-known general result. One can, though, remark that in the examples of Peternell-Sommese type “(universal family of lines on  $Y$ )  $\rightarrow Y$ ”, the inverse image of a  $(-1, 1)$ -line has a tendency to be connected, whereas the inverse image of a  $(0, 0)$ -line is certainly not connected. Indeed, it is observed in the literature that, on the threefolds as above (the cubic, the quadric line complex,  $V_5$ ), a line  $l$  is in the closure of the curve  $C_l = \{\text{lines intersecting } l \text{ but different from } l\}$  on the Hilbert scheme if and only if  $l$  is a  $(-1, 1)$ -line.

## 2. A Hilbert scheme argument

The previous considerations show that on our Fano threefold  $X$ , a disjoint union of conics degenerates flatly to a connected l.c.i. scheme. Recall the following classical example: if one degenerates a disjoint union of two lines in the projective space into a pair of intersecting lines, the pair of intersecting lines shall have an embedded point at the intersection. So if one wants the limit to be a connected l.c.i., this limit must be a double line. This suggests to ask if a similar phenomenon can occur in our situation, that is: can it be true that a connected l.c.i. limit of disjoint conics is necessarily a multiple conic?

In any case it is easily checked that, say, a connected limit of pairs of disjoint conics does not have to have embedded points when the two conics become reducible and acquire a common component. So even if a statement like this could be true, it is probably difficult to prove. In this paragraph we shall prove a weaker statement: the inverse image of a sufficiently general  $(-1, 1)$ -line is either a multiple conic, or supported on a union of lines.

Let  $T$  be the Hilbert scheme of lines on  $V_5$  and let  $\mathcal{M} \subset T \times V_5$  be the universal family. We have the “universal family of the inverse images of lines under  $f$ ”

$$\mathcal{S} = \mathcal{M} \times_{V_5} X \subset T \times X.$$

Since  $f$  is flat and  $\mathcal{M}$  is flat over  $T$ ,  $\mathcal{S}$  is flat over  $T$ .

Let  $H'$  be the Hilbert scheme of conics on  $X$ . Consider the irreducible components of  $H'$  which are relevant for our problem, that is, the components such that their sufficiently general points correspond to conics which are in the inverse image of a sufficiently general line on  $V_5$ . Denote by  $H$  the union of all such components.

For every point  $x \in H$ , the image of the corresponding conic  $C_x$  is a line. Indeed, “ $f(C)$  is a line” is a closed condition on conics  $C$  because  $f$  is a finite morphism (for  $f$  arbitrary, “ $f(C)$  is contained in a line” would be a closed condition on  $C$ ).

This allows to construct a morphism  $p : H \rightarrow T$  taking every conic to its image under  $f$ . Indeed,

$$\mathcal{L} = \{(C, f(x)) | x \in C, C \in H\} \subset H \times V_5$$

is a family of lines over  $H$ ; though apriori it is not clear that it is flat, this is a “well-defined family of algebraic cycles” in the sense of Kollar ([K], Chapter I) and so corresponds to a morphism from  $H$  to the Chow variety of lines on  $V_5$ , and this is the same as  $T$ .

We claim that  $p$  is finite. Indeed, it is clear that the only obstruction to the finiteness of  $p$  could be the existence of infinitely many double structures of arithmetic genus zero on some lines on  $X$  (“non-finiteness of the Hilbert-Chow morphism for the family of conics on  $X$ ”). This obviously happens if one considers conics in  $\mathbb{P}^3$  rather than conics on  $X$ . In our situation, however, this is impossible, and the Hilbert-Chow morphism is even one-to-one. Indeed, by [I], the normal bundle of a line in a prime Fano threefold is either  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , or  $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ , and there is the following

**Lemma 2.1** *Let  $l \subset X$  be a line on a prime Fano threefold. If  $N_{l,X} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , then there is no locally Cohen-Macaulay double structure of arithmetic genus 0 on  $l$ . If  $N_{l,X} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ , then such a structure is unique.*

*Proof:* All locally Cohen-Macaulay double structures on smooth curves in a threefold are obtained by a construction due to Ferrand (see for example [BF], or else [N] for details): if  $Y \subset V$  is a smooth curve on a smooth threefold, and  $\tilde{Y}$  is a double structure on  $Y$ , write  $L$  for  $\mathcal{I}_Y/\mathcal{I}_{\tilde{Y}}$ ; in fact  $L$  is a locally free rang-one  $\mathcal{O}_Y$ -module and  $\mathcal{I}_{\tilde{Y}}$  contains  $\mathcal{I}_Y^2$ . The double structure is thus determined by the natural surjection from the conormal bundle of  $Y$  in  $V$  to  $L$ , up to a scalar. Now take  $Y = l$ ,  $V = X$  and let  $L$  be as above; we have an exact sequence

$$0 \rightarrow L \rightarrow \mathcal{O}_{\tilde{l}} \rightarrow \mathcal{O}_l \rightarrow 0,$$

from which it is clear that  $p_a(\tilde{l}) = 0$  if and only if  $L = \mathcal{O}_{\mathbb{P}^1}(-1)$ . Now in the first part of our assertion, there is no non-trivial surjection from  $N_{l,X}^*$  to  $\mathcal{O}_{\mathbb{P}^1}(-1)$ , and in the second part, such a surjection is unique up to a scalar.

Note that we do not have to consider curves which are not locally Cohen-Macaulay, since, for example, the above argument shows that there are no higher genus locally Cohen-Macaulay double structures, and an embedded point decreases the genus.

Thus, for any  $t \in T$ ,  $p^{-1}(t)$  is a finite set  $\{h_1, \dots, h_k\}$ , and to each  $h_i$  there corresponds one conic  $C_i$  on  $X$ , mapped to  $l_t$  by  $f$ . The next step is to show that  $f$  and  $p$  “agree with each other”:

**Lemma 2.2** *Let  $t \in T$  be any point and  $l_t \in V_5$  be the corresponding line. Let  $h_1, \dots, h_k$  be the points of  $p^{-1}(t)$  and  $C_1, \dots, C_k$  the corresponding conics on  $X$ . Then the support of  $f^{-1}(l_t)$  is  $\bigcup_i C_i$ .*

*Proof:* Indeed, for a general  $t \in T$ , it is true:  $f^{-1}(l_t) = \bigcup_i C_i$ . For a special  $t \in T$ , choose a curve  $V \subset T$  through  $t$ , such that  $t$  is the only “non-general” point of  $V$  in the above sense, and let  $U = p^{-1}(V)$ . Denote by  $\mathcal{C}_U \subset U \times X$  the restriction to  $U$  of the universal family of conics over  $H$ . The support of the fiber over  $t$  of  $(p \times id)(\mathcal{C}_U) \subset V \times X$  is equal to  $\bigcup_i C_i$ . But the family  $\mathcal{S}|_V$  coincides with  $(p \times id)(\mathcal{C}_U)$  except at  $t$ .  $\mathcal{S}|_V$  being flat, it must be the scheme-theoretic closure of  $(p \times id)(\mathcal{C}_U)|_{V-\{t\}}$  in  $V \times X$ , and thus the support of  $\mathcal{S}|_V$  is  $(p \times id)(\mathcal{C}_U)$ , q.e.d.

Let now  $t \in T$  be a point corresponding to a sufficiently general  $(-1, 1)$ -line. We know that  $f^{-1}(l_t)$  is connected. Suppose that the number  $k$  from the Lemma is  $> 1$ , so that there are several conics in the  $Supp(f^{-1}(l_t))$ . Decompose the set of those conics into two disjoint non-empty subsets  $\Sigma_1$  and  $\Sigma_2$ .

**Proposition 2.3** *There exists a conic in  $\Sigma_1$  which has a common component with a conic in  $\Sigma_2$ ; in other words,  $(\bigcup_{C \in \Sigma_1} C) \cap (\bigcup_{C \in \Sigma_2} C)$  cannot be zero-dimensional.*

*Proof* Choose a suitable small 1-dimensional disc  $(V, 0)$  centered at  $t$ . The inverse image  $p^{-1}V$  is a disjoint union of two analytic sets  $U_1$  and  $U_2$  ( $U_i$  consists of points corresponding to conics near those of  $\Sigma_i$ ). Repeat the procedure of the previous lemma: consider the universal families  $\mathcal{C}_i$  of conics over  $U_i$  and their images  $\mathcal{S}_i = (p \times id)(\mathcal{C}_i) \subset V \times X$ . Let  $\mathcal{S}^0, \mathcal{S}_i^0$  denote the restriction of our families  $\mathcal{S}, \mathcal{S}_i$  to the punctured disc  $V^0 = V - \{0\}$ . The family  $\mathcal{S}^0$  is just the disjoint union of  $\mathcal{S}_i^0$ . Now take the closure of all those (as analytic spaces) in  $V \times X$ : the closure of  $\mathcal{S}^0$  is just  $\mathcal{S}|_V$ , by flatness, and the closure  $\mathcal{S}'_i$  of  $\mathcal{S}_i^0$  has the same support as  $\mathcal{S}_i$ , is contained in  $\mathcal{S}|_V$  and is flat over  $V$ . The fiber of  $\mathcal{S}'_i$  over 0, denoted  $S_i$ , is contained in the fiber  $S$  of  $\mathcal{S}$ , since the tensor multiplication preserves the surjectivity. So  $f^{-1}(l_t) = S$  contains  $S_1 \cup S_2$ . By construction,  $S_i$  are flat limits of disjoint unions of  $a_i$  conics and  $S$  is a flat limit of disjoint unions of  $a_1 + a_2$  ( $= \frac{m^2 \deg(X)}{10}$ ) conics.

If  $S_1$  and  $S_2$  do not have common components, then, since by flatness  $\deg(S) = \deg(S_1) + \deg(S_2)$ , this implies  $S = S_1 \cup S_2$ , because  $S$  is purely one-dimensional (being an inverse image of a line under a finite morphism). But then we can apply the exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \rightarrow \mathcal{O}_{S_1 \cap S_2} \rightarrow 0$$

and get a contradiction, since by flatness  $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S_1}) + \chi(\mathcal{O}_{S_2})$ ,  $S_1 \cap S_2$  is non-empty and it is zero-dimensional by assumption. Thus  $S_1$  and  $S_2$  must have common components, and, as  $S_i$  is supported on  $\bigcup_{C \in \Sigma_i} C$ , the Proposition is proved.

**Corollary 2.4** *In the situation as above,  $f^{-1}(l_t)$  is supported either on a single conic, or on a union of lines.*

Indeed, the proposition shows that if  $f^{-1}(l_t)$  contains more than one conic, then

any conic from  $f^{-1}(l_t)$  must have a common component with the rest of these conics, that is, it must be singular.

Some results from commutative algebra allow to prove a stronger (“local”) version of Proposition 2.3:

**Proposition 2.5** *In the situation of Proposition 2.3, through each intersection point  $P$  of  $\bigcup_{C \in \Sigma_1} C$  and  $\bigcup_{C \in \Sigma_2} C$  passes some common component of  $\bigcup_{C \in \Sigma_1} C$  and  $\bigcup_{C \in \Sigma_2} C$ .*

*Proof:* The family  $\mathcal{S}$  is flat over  $T$  which is smooth, and the fibers are l.c.i., thus locally Cohen-Macaulay. It follows ([EGA], 6.3.1, 6.3.5) that  $\mathcal{S}$  is locally Cohen-Macaulay, and that the same is true for the restriction of  $\mathcal{S}$  to any smooth curve in  $T$ . Suppose that Proposition 2.5 is not true for some intersection point  $P$ . Let  $x = (t, P) \in T \times X$  be the point corresponding to  $P$  in  $\mathcal{S}$ . Consider the restriction of  $\mathcal{S}$  to a general curve through  $t$ , and an analytic neighbourhood of  $x$  in this restriction. Clearly, if one removes  $x$ , this neighbourhood becomes disconnected: there are at least two branches corresponding to  $\text{Supp} \mathcal{S}_i$  as in Proposition 2.3. But this is impossible by Hartshorne’s connectedness ([H]), which implies that a connected Cohen-Macaulay neighbourhood remains connected if one removes a subvariety of codimension at least two.

**Remark 2.6** The argument of the Proposition is more or less the following: “if we have a disjoint union of certain smooth curves  $A$  and  $B$ , which degenerates flatly into a certain connected  $C$  in such a way that  $A$  and  $B$  do not acquire common components in the limit, then  $C$  will have embedded points at the intersection points of the limits of  $A$  and  $B$ , so this is impossible if we know that  $C$  is purely one-dimensional”. Examples show that one cannot say anything reasonable if one allows  $A$  and  $B$  to acquire common components. But in fact our “ $C$ ”, that is,  $f^{-1}(l_t)$ , is more than just purely one-dimensional: it is a locally complete intersection. I do not know if its being a flat limit of disjoint unions of conics can impose stronger restrictions on its geometry.

To illustrate how we shall apply this, let us handle the case when  $f^{-1}(l_t)$  is supported on a single conic.

**Proposition 2.7** *In this case  $X = V_{10}$  and  $f$  is a double covering.*

*Proof:* As the degree of the subscheme  $f^{-1}(l_t)$  of  $X$  is  $\frac{m^2 \deg(X)}{5}$ , this conic is of multiplicity  $\frac{m^2 \deg(X)}{10}$  in  $f^{-1}(l_t)$ . That is, the local degree of  $f$  near a general point of such a conic is also  $\frac{m^2 \deg(X)}{10}$ . Now this is the local degree of  $f$  along a certain divisor, because we have chosen the line  $l_t$  to be “sufficiently general among the  $(-1, 1)$



lines”: it varies in a one-dimensional family. This divisor is thus a component of the ramification divisor of  $f$ , and  $\frac{m^2 \deg(X)}{10} - 1$  is its ramification multiplicity.

Now the ramification divisor of  $f$  is an element of  $|\mathcal{O}_X(2m - 1)|$ , and so the local degree of  $f$  at its general point is at most  $2m$ , and if it is  $2m$ , then the ramification divisor is the inverse image of the surface covered by the  $(-1, 1)$ -lines and set-theoretically a hyperplane section of  $X$ . So we have:

$$\frac{m^2 \deg(X)}{10} \leq 2m, \quad m \deg(X) \leq 20,$$

and if the equality holds, then  $f$  is unramified outside the inverse image of the surface of  $(-1, 1)$ -lines. Also,  $\frac{m^2 \deg(X)}{10}$  must be an integer. The inequality thus only holds for  $\deg(X) = 10$  and  $m = 1$  (this is a double covering) or  $m = 2$  (in this case it is an equality), and for  $\deg(X) = 4$  and  $m = 5$  (also an equality). Let us exclude the last two cases. If  $f$  is unramified outside the inverse image of the surface of  $(-1, 1)$ -lines, then  $p$  is  $\frac{m^2 \deg(X)}{10}$ -to-one everywhere except over the conic parametrizing the  $(-1, 1)$ -lines on  $T = \mathbb{P}^2$ . It is thus a topological covering of the complement to this conic in  $T$ . But the latter is simply-connected; so that  $H$  has  $\frac{m^2 \deg(X)}{10}$  irreducible components and each one maps one-to-one on  $T$ . Notice that the number  $\frac{m^2 \deg(X)}{10}$  is superiour to three in both cases. But this is impossible. Indeed, on  $V_5$  one has only 3 lines through a general point; whereas, if  $H$  has  $k$  components, each component would give at least one conic through a general point of  $X$ . Those conics are mapped to *different* lines through  $f(x)$ , because they intersect; thus  $k \leq 3$ .

### 3. Proof of the Theorem

We have seen that the inverse image of a general  $(-1, 1)$ -line is supported either on one conic, or on a union of lines, and settled the first case in the end of the second section. Let us now settle the remaining case, using Proposition 2.5.

The following lemma is standart (and follows e.g. from the arguments of [M], Chapter 3):

**Lemma 3.1** *Let  $g : X_1 \rightarrow X_2$  be a proper morphism of complex quasiprojective varieties, which is finite of degree  $d$ . Suppose that  $X_2$  is smooth. Then the inverse image of any point  $x \in X_2$  consists of  $d$  points at most, and if there are exactly  $d$  points in the inverse image of all  $x \in X_2$ , then topologically  $g$  is a covering.*

Let  $H$  be as in the last section, and let  $\mathcal{C}$  be the universal family of conics over  $H$ . Each conic of  $H$  is contained in the inverse image of some line on  $V_5$ , and set-theoretically such an inverse image is a union of conics of  $H$ . Denote by  $D$  the surface covered by  $(-1, 1)$ -lines on  $V_5$ . Recall that through each point of the complement to  $D$  in  $V_5$  there are three lines, that  $D$  is a tangent surface to a rational normal sextic and that there are two lines (one  $(-1, 1)$ -line and one  $(0, 0)$ -line) through any point

of  $D$  away from this sextic and a single line through each point of the sextic. Since the inverse image of a general  $(0, 0)$ -line is a disjoint union of conics of  $H$ , there are three conics of  $H$  through a general point of  $X$ , and at least three through any point away from  $f^{-1}(D)$ . The natural morphism  $q : \mathcal{C} \rightarrow X$  is proper and finite of degree three. By the Lemma, there are exactly three conics of  $H$  through any point of  $X$  away from  $f^{-1}(D)$ .

Let  $l$  be a general  $(-1, 1)$ -line on  $V_5$ . Consider the case when  $Z = f^{-1}(l)$  is a set-theoretic union of degenerate conics  $C_1, \dots, C_k$  of  $H$ .

**Lemma 3.2**  *$Z$  contains a line which belongs to a single  $C_i$  (say  $C_1$ ).*

*Proof:* Suppose the contrary, that is, that any component of  $Z$  is contained in at least two conics of  $H$ . Through a general point  $x$  of this component there is at least one more conic of  $H$ , coming from the inverse image of the  $(0, 0)$ -line through  $f(x)$ . This implies that the morphism  $q : \mathcal{C} \rightarrow V_5$  is three-to-one outside an algebraic subset  $A$  of codimension at least two in  $X$ . That is,  $\mathcal{C} - q^{-1}(A)$  is, topologically, a covering of  $X - A$ . But  $X - A$  is simply-connected because  $X$  is Fano and thus simply-connected. This means that  $\mathcal{C}$  is reducible, consists of three components and each of them maps one-to-one to  $X$ . Since  $X$  is smooth, it must be isomorphic to each of those components (by Zariski's Main Theorem). But this is impossible because the components are fibered in conics and  $X$  has cyclic Picard group.

Before continuing our argument, let us recall some well-known facts on lines on prime Fano threefolds ([I]). Lines on our Fano threefold  $X$  are parametrized by a curve, which may of course be reducible or non-reduced. Its being reduced or not influences the geometry of the surface covered by lines on  $X$ . Namely, if a component of the Hilbert scheme of lines on  $X$  is reduced, then the natural morphism from the correspondent component of the universal family to  $X$  is an immersion along a general line; and there is a classical computation ([I], [T]) which says that if its image  $M$  is an element of  $|\mathcal{O}_X(d)|$ , then a general line of  $M$  intersects  $d + 1$  other lines of  $M$ . If a component of the Hilbert scheme of lines is non-reduced, then the surface  $M$  covered by the corresponding lines is either a cone (but this can happen only on a quartic), or a tangent surface to a curve. One knows only one explicit example of a Fano threefold as above such that the surface covered by lines on it is a tangent surface to a curve, it is constructed by Mukai and Umemura (having been overlooked by Iskovskih) and has degree 22. The surface itself is a hyperplane section of this threefold and its lines never intersect.

The following Proposition, due to Iliev and Schuhmann, is the main result of [IS] slightly reformulated:

**Proposition 3.3** *Let  $X$  be a prime Fano threefold,  $\mathcal{L}$  a complete one-dimensional family of lines on  $X$  and  $M$  the surface on  $X$  covered by lines of  $\mathcal{L}$ . If  $X$  is different from the Mukai-Umemura threefold, then a general line of  $\mathcal{L}$  intersects at least one*

other line of  $\mathcal{L}$ .

*An outline of the proof:* If not, then, by what we have said above, the surface  $M$  must be a tangent surface to a curve. Studying its singularities, Iliev and Schuhmann prove that it must be a hyperplane section of  $X$ . Then they show, by case-by-case analysis (of which certain cases appear already in [A]), that the only prime Fano threefold containing a tangent surface to a curve as a hyperplane section, is the Mukai-Umemura threefold.

“Lines contained in a single  $C_i$ ” cover a divisor on  $X$  as  $Z$  varies (this is the branch divisor of  $q$ ). Since  $(-1, 1)$ -lines on  $V_5$  never intersect, Proposition 3.3 implies that if  $X$  is not the Mukai-Umemura threefold, then in  $Z$  there are at least two lines contained in a single conic (say,  $l_1 \subset C_1$  and  $l_2 \subset C_2$ ), and that they intersect, say at the point  $P$ . Notice that  $C_1$  is necessarily different from  $C_2$ : otherwise we get a contradiction with Proposition 2.5 by considering  $\Sigma_1 = \{l_1 \cup l_2\}$ ,  $\Sigma_2$  the set of all the other  $C_i$  and the intersection point  $P$ .

**Claim 3.4** Both  $C_1$  and  $C_2$  are pairs of lines intersecting at the point  $P$ , and  $Z$  is supported on  $C_1 \cup C_2$ . Thus  $Z$  is, set-theoretically, the union of three or four lines through  $P$ .

*Proof:*

1) If  $C_1$  is a double line, we get a contradiction with Proposition 2.5 by considering  $\Sigma_1 = \{C_1\}$  and the point  $P$ ; the same is true for  $C_2$ .

2) Let  $C_1 = l_1 \cup l'_1$ . If  $l'_1$  does not pass through  $P$ , we get the contradiction in the same way, thus  $P \in l'_1$ . Also,  $P \in l'_2$ , where  $C_2 = l_2 \cup l'_2$ .

3) There are two possibilities:

a) If  $l'_1 \neq l'_2$ , then there must be another conic from  $Z$  through  $P$ , containing  $l'_1$ . Indeed, otherwise we again get a contradiction with Proposition 2.5. In the same way, there is a conic from  $Z$  through  $P$  which contains  $l'_2$ . In fact it is the same conic, because otherwise there are at least four conics through  $P$ , contradicting Lemma 3.1. Denote it by  $C_3$ . No other conic from  $Z$  passes through  $P$ . So  $C_3 = l'_1 \cup l'_2$ , and  $l'_1, l'_2$  are not contained in conics others than  $C_1, C_2, C_3$ .

b) If  $l'_1 = l'_2$ , then no other conic from  $Z$  contains this line (otherwise through its general point there will pass at least four conics from  $H$ , the fourth one coming from the inverse image of the correspondent  $(0, 0)$ -line).

4) Now the union  $C_1 \cup C_2 \cup C_3$  in the case a), resp. the union  $C_1 \cup C_2$  in the case b), cannot have any points in common with the other components of  $Z$ ; otherwise, taking  $\Sigma_1 = \{C_1, C_2, C_3\}$ , resp.  $\Sigma_1 = \{C_1, C_2\}$ , we obtain a contradiction with Proposition 2.5. But  $Z$  is connected, so  $Z$  is supported on the lines  $l_1, l'_1, l_2, l'_2$ , q.e.d.

We are now ready to finish the proof of the theorem stated in the introduction.

*Proof of the theorem:* If  $X$  is the Mukai-Umemura threefold, then the lines on  $X$  never intersect at all, so that  $f^{-1}(l)$  must be supported on a single conic. Proposition 2.7 shows that a morphism from  $X$  to  $V_5$  is impossible. (It should be, however, said at this point that the paper [HM] contains a better proof of the non-existence of morphisms from the Mukai-Umemura threefold onto any other smooth variety, besides  $\mathbb{P}^3$ !).

If  $X$  is not the Mukai-Umemura threefold and  $f^{-1}(l)$  is not supported on a single conic, then we know by Claim 3.4 how  $f^{-1}(l)$  looks. Remark that  $f^{-1}(D)$  is a reducible divisor: it has two components, one swept out by the lines  $l_1$  and  $l_2$  as  $Z$  varies, another by  $l'_1$  and  $l'_2$ . Neither component is a hyperplane section: indeed, if a hyperplane section of  $X$  is covered by lines, then it is either a cone (impossible in our situation), or a general line intersects two other lines on the surface by the classical computation from [T] mentioned above, since a hyperplane section cannot be a tangent surface to a curve by [IS]. Let  $k$  be the multiplicity of the component corresponding to  $l_i$  and  $k'$  be the multiplicity of the component corresponding to  $l'_i$ . As  $f^*(D)$  is a divisor from  $|\mathcal{O}_X(2m)|$ ,  $k + k' \leq m$ . At the same time,  $Z$  must be of degree  $\frac{m^2 \deg(X)}{5}$ , and thus  $2k + 2k' = \frac{m^2 \deg(X)}{5}$ , so  $m^2 \deg(X) \leq 10$ , leaving the only possibility  $m = 1$ ,  $\deg(X) = 10$ .

#### 4. Concluding remarks

In this section, we shall make a further (minor) precision on Theorem 3.1 from [A].

In that theorem, it was proved that if  $X, Y$  are Fano threefolds with Picard number one and very ample generator of the Picard group,  $X$  is of index one,  $Y$  is of index two different from  $V_5$  (that is,  $Y$  is a cubic or a quadric line complex), and  $f : X \rightarrow Y$  is a surjective morphism, then  $f$  is a “projection”, that is,  $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(1)$ . The argument of the theorem also worked for  $Y$  a quartic double solid, whereas there were some problems (hopefully technical ones) for  $Y$  a double Veronese cone and for  $X$  not anticanonically embedded.

Even in the “good” cases, the theorem proves a little bit less than one would like; that is, we want  $f$  to be a double covering and we prove only that  $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(1)$ . This still leaves the following additional possibilities:

- (1) If  $Y$  is a cubic,  $X$  can be  $V_{12}$ ,  $\deg(f) = 4$  ( $X$  cannot be  $V_{18}$  because of the Betti numbers:  $b_3(V_{18}) < b_3(Y)$ );
- (2) If  $Y$  is an intersection of two quadrics,  $X$  can be  $V_{16}$ ,  $\deg(f) = 4$  (here  $V_{12}$  is impossible since in this case the inverse image of a general line would consist of  $3/2$  conics).

The first possibility can be excluded by using an inequality of [ARV]: it says that for a finite morphism  $f : X \rightarrow Y$  and a line bundle  $L$  on  $Y$  such that  $\Omega_Y(L)$  is globally generated,  $\deg(f)c_{top}\Omega_Y(L) \leq c_{top}\Omega_X(f^*L)$ , so, for  $X$  and  $Y$  of dimension three,  $\deg(f)(c_3(\Omega_Y) + c_2(\Omega_Y)L + c_1(\Omega_Y)L^2)$  must not exceed  $c_3(\Omega_X) + c_2(\Omega_X)f^*L + c_1(\Omega_X)f^*L^2$ .

Consider the situation of (1): we may take  $L = \mathcal{O}_Y(2)$ , and we know that  $c_3(\Omega_Y) = 6$  and  $c_3(\Omega_X) = 10$ . Using the equalities  $c_2(X)c_1(X) = c_2(Y)c_1(Y) = 24$ , we arrive at  $4(6 + 24 - 24) \leq 10 + 48 - 48$ , which is false. So the case (1) cannot occur.

This inequality does not work in the case (2): indeed, now  $c_3(\Omega_Y) = 0$ ,  $c_3(\Omega_X) = 2$  and the inequality reads as follows:  $4(0 + 24 - 32) \leq 2 + 48 - 64$ , so does not give a contradiction. However we can rule out this case by our connectedness argument. Indeed, the inverse image of a general  $(-1, 1)$ -line is connected (Remark 1.4) and the inverse image of a general  $(0, 0)$ -line consists of two disjoint conics. The results of Section 2 apply, of course, to our situation; it follows that the inverse image of a general  $(-1, 1)$ -line is either a double conic, or a union of two reducible conics which have a common component. In both cases, it is clear that the ramification locus of  $f$  projects *onto* the surface covered by  $(-1, 1)$ -lines. But the ramification divisor is a hyperplane section of  $V_{16}$ , and thus can project onto a surface from  $|\mathcal{O}_Y(4)|$  at most. Whereas it is well-known (and follows for example from the results in [GH], Chapter 6) that the surface covered by  $(-1, 1)$ -lines on  $Y$  is an element of  $|\mathcal{O}_Y(8)|$ .

All this put together gives the following

**Theorem 4.1** *Let  $X, Y$  be smooth complex Fano threefolds of Picard number one,  $X$  of index one,  $Y$  of index two. Assume further that the ample generators of  $\text{Pic}(X)$  and  $\text{Pic}(Y)$  are very ample. Then any morphism from  $X$  to  $Y$  is a double covering.*

I would like to mention that the verification of this statement without the very ampleness hypothesis amounts to a very small number of particular cases; for instance, if  $Y$  is a double Veronese cone, then already the formula of [ARV] combined with the knowledge of Betti numbers implies that for any morphism  $f : X \rightarrow Y$  with  $X$  Fano of index one with cyclic Picard group,  $\deg(f) = 2$  and  $X$  is a sextic double solid. It seems that one could be able to work out the remaining cases without any essentially new ideas.

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